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Best Monotone Approximation by Reciprocals of Polynomials

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Recently there has been a great deal of interest in approximating functions by polynomials, splines, or rational functions subject to constraints. In this context one asks: do best approximations exist? If so, how are they characterized and are they unique? These questions are in general more difficult with constraints than without them. For example, Loeb [2] shows that existence of best uniform rational approximations fails when the constraint is interpolation at a finite number of points.

In this note we consider uniform approximation of functions $f \in C[a, b]$ by rational functions $r \in R_n^0$ which are non-decreasing on $[a, b]$. This will be called the problem of best monotone approximation by reciprocals of polynomials. The problem was proposed by Taylor [4] as a reasonable first step toward the much more difficult problem of monotone approximation from R_n^m. The corresponding problem for polynomials was solved by Lorentz and Zeller [4]. Many of the techniques in this note are taken from their work.

Let π_k be the space of algebraic polynomials of degree $\leqslant k$ and

$$R_m^{n^*} = \{r = p/q \colon p \in \pi_n, q \in \pi_m, q(x) > 0 \text{ on } [a, b] \text{ and } r'(x) \geqslant 0 \text{ on } [a, b]\}.$$

Let $\|\cdot\|$ denote the uniform norm on $[a, b]$. The existence of an $r^* \in R_m^{n^*}$ of best uniform approximation to $f \in C[a, b]$, that is, of an $r^* \in R_m^{n^*}$ such that

$$\|f - r^*\| = \inf_{r \in R_m^{n^*}} \|f - r\|,$$

follows from a standard argument (see Cheney [1, p. 154]). We note that existence will hold whenever the constraint is of the form $M_j(x) \geqslant r^{(j)}(x) \geqslant m_j(x)$, $x \in [a, b]$, $j = 0, 1, \ldots, k$ with all the functions M_j, m_j in $C[a, b]$, and at

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least one $r \in R_m^n$ satisfying the constraint. The continuity of the functions m_j, M_j guarantees that in taking the usual pointwise, except at a finite number of points, limit to prove existence we do not go outside the feasible region. This is precisely where existence fails in the case of Lagrange interpolatory constraints (see Loeb [2]).

The following characterization of best uniform approximations from $R_n^{0^*}$ holds:

THEOREM 1. *Let* $n \geqslant 1$. $f \in C[a,b] \backslash R_n^{0^*}$ *and* $\max_{[a,b]} f(x) \neq -\min_{[a,b]} f(x)$. *An element* $r = \sigma/p \in R_n^{0^*}$, *where* $\sigma \in \{\pm 1\}$ *and* $p(x) > 0$ *on* $[a,b]$, *is a best approximation to f iff there is no* $q \in \pi_n$ *with the following properties*:

$$\operatorname{sign}(q(x)) = \operatorname{sign}(f(x) - r(x))$$

for all x in the set

$$A(f,r) = \{x \in [a,b] : |f(x) - r(x)| = \|f - r\|\}, \tag{1}$$

and $q'(x) > 0$ *for all x in the set*

$$B(r) = \{x \in [a,b] : r'(x) = 0\}.$$

Proof of Necessity. Case 1. $\max_{[a,b]} f(x) > -\min_{[a,b]} f(x)$. In this case it is clear that the positive constant $r = (\max_{[a,b]} f(x) + \min_{[a,b]} f(x))/2$ does better than any non-positive member of $R_n^{0^*}$. Thus if r is any best approximation, $r = 1/p$ for some $p \in \pi_n$ positive on $[a,b]$.

Suppose there is a $q \in \pi_n$ with the properties (1). Consider

$$r_\lambda = 1/(p - \lambda q).$$

Well-known arguments (see Cheney [1, pp. 159–160]) show that there exists a $\lambda_1 > 0$ so that $\lambda_1 \geqslant \lambda > 0$ implies r_λ is positive on $[a,b]$ and $\|f - r_\lambda\| < \|f - r\|$. We will now show that there exists a λ, $\lambda_1 \geqslant \lambda > 0$ such that $r'_\lambda(x) \geqslant 0$ on $[a,b]$. Set $C = \{x \in [a,b] : q'(x) \leqslant 0\}$. By the continuity of q', C is a compact set containing no points of B. Hence by the continuity of $p'(x)$ and compactness there exists a d such that $p'(x) \leqslant d < 0$ for all $x \in C$. Choose λ, $\lambda_1 \geqslant \lambda > 0$, so small that $\|\lambda q'\| < -d$ then $p'(x) - \lambda q'(x)$ will be negative on $[a,b]$. Hence $r'_\lambda(x) > 0$ on $[a,b]$. But then $r_\lambda \in R_n^0{}^*$ and is a better approximation to f than r. Contradiction. Thus there cannot exist a $q \in \pi_n$ with the properties (1).

Case 2. $\max_{[a,b]} f(x) < -\min_{[a,b]} f(x)$. In this case any best approximation has the form $r = -1/p$, where $p \in \pi_n$ is positive on $[a,b]$. We can obtain a contradiction by assuming $q \in \pi_n$ has the properties (1) and considering $r_\lambda = -1/(p + \lambda q)$.

Proof of Sufficiency. *Case* 1. $r = 1/p$, $p(x) > 0$, $x \in [a, b]$. Suppose r is not a best approximation of f. Then there is a positive $\tilde{r} = 1/\tilde{p}$ which is better. For if $\max_{[a,b]} f(x) > -\min_{[a,b]} f(x)$ there is a positive best approximation. If, on the other hand, $\max_{[a,b]} f(x) \leqslant -\min_{[a,b]} f(x)$, then $\|f - r\| > -\min_{[a,b]} f(x)$. Hence for a large enough positive constant C, $\|f - 1/C\| < \|f - r\|$.

Write

$$\tilde{r} = \frac{1}{p - (p - \tilde{p})} = \frac{1}{p - \bar{p}}.$$

Since

$$(f - \tilde{r})(x) < (f - r)(x) \qquad \text{when} \quad (f - r)(x) = \|f - r\|,$$

and

$$(f - \tilde{r})(x) > (f - r)(x) \qquad \text{when} \quad (f - r)(x) = -\|f - r\|,$$

it follows that

$$\operatorname{sign}(\bar{p}(x)) = \operatorname{sign}((f - r)(x)) \qquad \text{for} \quad x \in A.$$

Also

$$\tilde{r}'(x) = -\frac{(p - \bar{p})'(x)}{(p - \bar{p})^2(x)} \geqslant 0, \qquad x \in [a, b],$$

so that $\bar{p}'(x) \geqslant 0$ for $x \in B$. Since A is compact and $\bar{p}$ continuous there exists a $\delta > 0$ so that $|\bar{p}(x)| > \delta$ for $x \in A$. Hence for sufficiently small $\varepsilon > 0$ the function

$$q(x) = \bar{p}(x) + \varepsilon x$$

will have the properties (1). Thus, by contraposition, if there is no q with the properties (1) r must be a best approximation.

Case 2. $r = -1/p$, $p(x) > 0$, $x \in [a, b]$. In this case $\tilde{r}$ is chosen as a better negative approximant and $\bar{p}$ defined by $\tilde{r} = -1/(p + \bar{p})$. The rest of the details are the same as those in case 1. ∎

The preceding characterization theorem is strikingly similar to that for best monotone approximation from π_n. In that case $p \in P_k = \{s \in \pi_n : s^{(k)}(x) \geqslant 0 \text{ on } [a, b]\}$ is a best approximation iff there is no $q \in \pi_n$ with the properties

$$\operatorname{sign}(q(x)) = \operatorname{sign}((f - p)(x))$$

for all x in $A(f, p)$ and

$$q^{(k)}(x) > 0 \qquad \text{for all } x \text{ in } B(p).$$

[Here $B(p) = \{x \in [a, b]: p^{(k)}(x) = 0\}$.] Lorentz and Zeller [3] proved uniqueness from characterization, in the polynomial case, by using methods of Birkhoff interpolation. An analogous arguments works for $R_n^0{}^*$.

Let the characterization theorem apply to f and $r = \sigma/p \in R_n^0{}^*$ be a best approximation of f. Let m be the number of points of A, l be the number of points of B, and e be the number of points of B among a and b. If $m < \infty$, let $x_1,..., x_m$ be the members of A in ascending order. If $l < \infty$, let $y_1,..., y_l$ be the members of B. The following theorem is the analogue of [3, Theorem 9]. The terminology used in the proof is standard in Birkhoff interpolation theory. It coincides with that of [3, p. 11].

THEOREM 2. *Let the characterization theorem apply to f and $r = \sigma/p \in R_n^0{}^*$, $n \geqslant 1$, be a best approximation to f. Then*

$$m + 2l - e \geqslant n + 2.$$

Proof. Assume on the contrary $m + 2l - e \leqslant n + 1$. Consider the Birkhoff interpolation problem (for polynomials q of degree $\leqslant m + 2l - e - 1$)

$$q(x_i) = a_i, \qquad i = 1,..., m, \qquad q'(y_j) = b_j, \qquad j = 1,..., l.$$

Add to these conditions,

$$q(y_j) = c_j, \qquad a < y_j < b$$

with arbitrary data c_j unless $y_j = x_i$ in which case we take $c_j = a_i$.

The incidence matrix E corresponding to this interpolation problem has $k \leqslant m + 2l - e$ non-zero entries, all of which occur in the first two columns. Also from the definition of A $(=A(f, r))$ and the continuity of $f - r$, $m \geqslant 1$. Hence E satisfies the Polya condition. The only points x at which a condition on q' can occur without a corresponding condition on q are a and b. Hence E contains no supported sequences. It follows from the theorem of Atkinson and Sharma that the matrix E is free. In particular, there is a polynomial q of degree $\leqslant k - 1 \leqslant m + 2l - e - 1 \leqslant n$ with $q(x_i) = \operatorname{sign}((f - r)(x_i))$ for $i = 1,..., m$, and $q'(y_j) = 1$ for $j = 1,..., l$. This contradicts the characterization theorem. ∎

THEOREM 3. *Let $f \in C[a, b]$. There is a unique best approximation, r, of f in $R_n^0{}^*$.*

Proof. Existence has already been discussed. It remains to prove uniqueness.

If $f \in R_n^0{}^*$, then clearly $r = f$ is the unique best approximation. If $n = 0$, $R_n^0{}^* = \pi_0$ and the result is classical. If $\max_{[a,b]} f(x) = -\min_{[a,b]} f(x)$, then $r(x) \equiv 0$ does better than any other member of $R_n^0{}^*$. This since all other functions in $R_n^0{}^*$ are either positive throughout $[a, b]$ or negative throughout $[a, b]$.

It remains to discuss the case when the characterization theorem applies to f. Let $r_1 = \sigma/p_1$, $r_2 = \sigma/p_2$ be two best approximations. Let $p_0 = (p_1 + p_2)/2$. Then from the inequality

$$\min(p_1(x), p_2(x)) \leqslant p_0(x) \leqslant \max(p_1(x), p_2(x))$$

it follows that $r_0 = \sigma/p_0$ is also a best approximation. Note that, $x \in A(f, r_0)$ implies $p_0(x) = p_1(x) = p_2(x)$, and $x \in B(r_0)$ implies $x \in B(r_1)$ and $x \in B(r_2)$. From the equations

$$r' = -\frac{\sigma p'}{p^2}, \qquad r'' = \frac{\sigma(2(p')^2 - p''p)}{p^3},$$

we see $p_i'(y) = p_i''(y) = 0$ for $y \in B(r_1) \cap (a, b)$ and $p_i'(y) = 0$ for $y \in B(r_i) \cap \{a, b\}$. Consider $D = p_0 - p_1$. From above $D(x) = 0$ for $x \in A = A(f, r_0)$, $D'(y) = 0$ for $y \in B = B(r_0)$ and $D''(y) = 0$ for $y \in B \cap (a, b)$.

We proceed to count the zeros of D'. Firstly at the $l - e$ points of $B \cap (a, b)$ D' has double zeros. At the e points of B among a and b, D' has simple zeros. So far on each interval (x_i, x_{i+1}) between two zeros of D our count includes only double zeros of D'. But on such an interval D' is either identically zero or has at least one zero of odd multiplicity. Thus D' has at least $m - 1$ more zeros, making a total of at least $2l - e + m - 1$. By Theorem 2 this is $\geqslant n + 1$. Hence D' is identically zero. Since $m \geqslant 1$, D is also identically zero. Thus $p_0 = p_1$ implying $p_1 = p_2$ and finally $r_1 = r_2$. ∎

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